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Note on bound states and the Bekenstein bound

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Abstract: In this brief note we draw attention to examples of quantum field theories which may hold interesting lessons for attempts to devise a precise formulation of the Bekenstein bound. Our comments mirror the recent results of Bousso [12] indicating that the species problem remains an issue for precise formulations of this bound.

Keywords: Nonperturbative Effects, Black Holes.

Contents

1. Introduction

There has been much discussion in the literature of the idea that quantum systems may be subject to certain fundamental bounds relating their entropy (S) to their size (measured in terms of a radius R or an enveloping area), and perhaps to their energy (E). Such proposed bounds include the Bekenstein bound $S < \alpha RE$ [1, 2], the holographic bound $S < A/4\ell_p^2$ [3, 4], and the more subtle Causal [5] and Covariant [6] Entropy Bounds. Such bounds were originally motivated by considerations of black hole thermodynamics [1]–[4]. Though this motivation has been criticized by various authors [7]–[10], the proposed bounds remain interesting topics of discussion and investigation.

The Covariant Entropy Bound represents a refinement of the holographic bound as, at least when spacetime can be treated classically, it gives a precise definition of what is meant by the area A. Similarly, the parameters playing the role of size and energy for the Causal Entropy Bound are well-defined in this context, though the same is not true of the original holographic bound. It is also of interest to study whether a more precise conjecture can be found to replace the Bekenstein bound $S < \alpha RE$. This was explored in two recent papers [11, 12] by Bousso. The Bekenstein bound is unique among those above in that it does not involve the Planck length. It may therefore be conjectured to hold in ordinary field theories, without considering coupling to gravity. This is advantageous for testing the bound, as we have more knowledge as to which such theories exist than we do when gravity is considered. An alternate interpretation of the Bekenstein bound is that, although it does not explicitly refer to the Planck length, it should apply only to field theories which can in principle be consistently coupled to the gravitational field. We shall have little to say here about this more restrictive conjecture.

In [11] (following Bekenstein [13, 14, 15]) it was argued that a precise version of this conjecture might apply to arbitrary quantum field theories. In particular, it was argued that a more precise formulation might be able to handle the so-called 'species problem', referring to the fact that naive interpretations of the bound $S < \alpha RE$ (where α is a fixed constant of order 1) are readily violated in any theory containing a large number of fields. A simple example arises from a one-particle wavepacket state in a theory of N massless scalar fields. Such a state has $RE \sim 1$, but $S \sim \ln N$. Thus, the most naive interpretation of the Bekenstein bound is violated.

Bekenstein has long argued that the bound should not apply to such wavepacket states (which will eventually spread out in space), but only to 'complete systems' [13, 14, 15] which are truly confined to a finite region and that one should include contributions from the energy of any 'walls' used to hold the system together. It is here that some cleverness is needed to make this statement precise since in flat spacetime, even if walls are introduced, the full system (including the walls) will necessarily possess an overall center of mass degree of freedom which will be unconfined and will eventually spread out across all of space. Thus, it is not clear in what sense any sub-system of the universe is truly 'complete' in this sense.

The final section of [11] suggested that one should simply disregard the overall center of mass degree of freedom and instead consider 'bound states,' with the size R being the width of the bound state. Following [11], we shall not yet be too precise about how this width is defined. We also note that another proposal was explored in [12], in which the conjecture was made precise in the context of discrete light cone quantum field theory, where the size is controlled by the size of a compact direction in the spacetime. However, it was noted in [12] that a large number of species can violate the bound as easily in this second context as in the naive example above.

Here we consider the 'bound state' proposal of [11] to remove the center of mass degree of freedom and define the resulting quotient to be the set of bound states. One may then test bounds of the form $S < \alpha RM$, where M is the mass of the bound states. Even in this context one may quickly construct counter-examples. First, consider again a theory of N free massless scalar particles. The above quotient of the one-particle Hilbert space leaves an N-dimensional vector space corresponding to particle flavor. But $S = \ln N$ and $M = 0$ clearly violates the bound, and at finite M the bound is violated for large enough N. This counter-example was also pointed out in [12]. A similar trivial violation may be constructed from any theory having a clear 'bound state' (say, QCD with its hadrons) and then considering a lagrangian built from a large number N of mutually non-interacting copies of this system.

Now, while the examples above are counter-examples in the technical sense, they appear to be somewhat trivial. This triviality might be taken as an indication that, with a bit of refinement, the bound state version of the Bekenstein bound could be made more robust. Our purpose below is to point out less trivial counter-examples in which the states appear at some intuitive level to all be 'bound states held together by the same force' — though the existence of dual formulations again raises the question of to what extent 'bound states' are fundamentally different from any other sort of state. Our counter-examples concern $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric $\mathrm{SU}(N_c)$ gauge theory, where the degrees of freedom are under some control (see for example [16]) in the infrared limit.

2. Examples: $\mathcal{N} = 1$ and $\mathcal{N} = 2$ $\mathrm{SU}(N_c)$ gauge theory with fundamental matter

Let us consider an $\mathrm{SU}(N_c)$ gauge theory in 3+1 dimensions with matter in the fundamental representation. The infrared behavior depends on the number N_f of matter multiplets Q and $\tilde{Q}$ (see for example [16]). Quite generally, if $N_f < 3N_c$ among the low energy degrees of freedom one finds, in some description, N_f^2 mesons $\mathcal{M} = Q\tilde{Q}$ and baryons, which are composites of the high energy matter fields. Of particular interest to our discussion is the situation $\frac{3}{2}N_c < N_f < 3N_c$ in which the theory has a nontrivial infrared fixed point; i.e., the theory flows to a conformal field theory in the infrared.[1] Since $\mathcal{M}$ are chiral fields, the superconformal algebra relates their dimension to their $\mathcal{R}$ charge through

$$\mathcal{D}(\mathcal{M}) = \frac{3}{2}\mathcal{R}(\mathcal{M}) = 3\frac{N_f - N_c}{N_f}\,, \tag{2.1}$$

which is less than the sum of the dimensions of the constituents, and thus one can think of the mesons as bound states.

Let us now consider the consequences of conformal invariance in the IR as relates to the proposed bound. When probed at finite energies, the mesons will be bound states of some finite size $R \geq 0$ and mass $M \geq 0$. When probed at very low energies E, it is natural that the product ER vanishes in the limit $E \to 0$. Thus, when considered from the point of view of the low energy theory (which naturally measures distance in units of $1/E$), the bound states have effectively zero size. In fact, this result is implied directly by conformal invariance, as the only conformally invariant possible values of R are zero or infinity. On the other hand, infinite meson size would not be consistent with a description of the mesons through local operators. Similarly, conformal invariance tells us that the mass of the bound states must vanish exactly; i.e., $M = 0$.

We may now note that for *any* N_f, N_c in our allowed range, counting the meson degrees of freedom as bound states with $E = M = 0$ violates $S < \alpha RE$ in the sense described above, even though the number of mesons is sometimes lower than the number of fundamental fields (for $\frac{3}{2}N_c < N_f < 2N_c$). Furthermore, even if one changes the rules and requires that the mesons be placed in wavepackets with $E \sim 1/R$, the bound is readily violated at large N_c, N_f.

In the limit $N_f \searrow \frac{3}{2}N_c$ the dimension of the meson fields becomes unity which leads to them being interpreted as free fields. We thus find the 'species problem' in a naively interacting field theory. One might be tempted to discard this limiting case based on the existence of a dual formulation in which the theory is free; it is however not clear why one should do this at intermediate energies.

Another class of examples in the direction described above is provided by $\mathcal{N} = 2$ theories with $\mathrm{SU}(N_c)$ gauge group and $N_f = 2N_c$ fundamental matter fields. Unlike the $\mathcal{N} = 1$, only this matter content yields a conformal theory, as can be seen from the perturbative β-function $\beta = 2N_c - N_f$. The relation between the $\mathrm{U}(1)_R$ charge and the dimension

$$\mathcal{D}(\mathcal{M}) = \frac{3}{2}\mathcal{R}(\mathcal{M}) = 2 \tag{2.2}$$

does not immediately imply that the mesons can be interpreted as bound states. It is not hard to see however that a small supersymmetry-breaking deformation (e.g. giving a nonvanishing mass to the scalars in the $\mathcal{N} = 2$ vector multiplet) leads to a decrease in

[1] The cases with no conformal invariance are quite hard to analyze because the masses of the low energy degrees of freedom depend on the Kähler potential and are not under control.

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the dimension of the meson fields and thus implies that they can be interpreted for our purposes as marginal bound states. The consequences of an infrared conformal fixed point are similar to the $\mathcal{N} = 1$ case, so we will not repeat them. As before, they lead to a violation of the entropy bound.

3. Brief discussion

We have drawn attention to a set of models which contain a large number of massless states associated with composite operators of more fundamental fields. In this sense, these states are bound and would violate a 'bound state version' of a Bekenstein bound. In fact, even a single massless bound state violates a strict interpretation of this bound. We therefore note that proponents of a bound-state version of a Bekenstein-like bound must advocate either 1) application of such bounds only to theories which can be consistently coupled to gravity or 2) a notion of bound state for which the above theories fail to qualify. Note that option (1) would be ruled out if a convincing theory of the above models coupled to gravity could be found. We mention in passing that similar results hold for SU(n) gauge theory with $m < n$ massless fundamental fermions and $\mathcal{N} = 2$ supersymmetry in 3+1 dimensions which are confining rather than conformal. Here the mass of mesons can be tuned to be arbitrarily small while taking the confinement energy scale to infinity as fast as one likes, and thus presumably keeping the size of the meson bound states small.

Let us return briefly to the actual context discussed in [11], in which Bousso attempted to study the confinement of degrees of freedom to a fixed region of space through the use of an external potential. The argument in [12] was that an attempt to use a single potential to confine a large number of species inevitably leads to large radiative corrections, over which one has little control. The bound states in the models above are of a similar nature, as the gluons couple equally to each of the N_f flavors of Q and $\tilde{Q}$, though now supersymmetry *does* allow one to retain some control over the analysis. In the context of such bound states one finds that degrees of freedom (the relative motion of the constituents) can in fact be confined without great cost in energy. This suggests that if tools could be found to make the analysis tractable, external potential problems of the sort studied in [11] could also lead to large numbers of states localized in a region of fixed size. As usual, we expect that strongly coupled quantum field theories are capable of all manner of surprising behaviors not immediately obvious from their perturbative description.

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